The LNM Institute of Information Technology, Jaipur

M.Sc. (MATH)
Admission Entrance Exam, 2018

Time: 2 Hours　　Date: 16/07/2018　　Maximum Marks: 100

There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, A, B and C. All sections are compulsory. Questions in each section are of different types.
Section - A contains a total of 30 Multiple Choice Questions (MCQ). Each MCQ type question has four choices out of which only one choice is the correct answer. Questions Q.1 - Q.30 belong to this section and carry a total of 50 marks. Q.1 - Q.10 carry 1 mark each and Questions Q.11 - Q.30 carry 2 marks each. Wrong answer will result in NEGATIVE marks. For all 1 mark questions, 1/3 marks will be deducted for each wrong answer. For all 2 marks questions, 2/3 marks will be deducted for each wrong answer.
Section - B contains a total of 10 Multiple Select Questions (MSQ). Each MSQ type question is similar to MCQ but with a difference that there may be one or more than one choice(s) that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.1-Q.10 belong to this section and carry 2 marks each with a total of 20 marks. There is NO NEGATIVE and NO PARTIAL marking provisions.
Section - C contains a total of 20 Numerical Answer Type (NAT) questions. Questions Q.1-Q.20 belong to this section and carry a total of 30 marks. Q.1-Q.10 carry 1 mark each and Questions Q.11-Q.20 carry 2 marks each. There is NO NEGATIVE marking.

Section A

1. Which of the following binary operation is not associative?
   (a) the operation $*$ on $\mathbb{Q} - \{0\}$ defined by $a * b = \frac{a}{b}$.
   (b) the operation $*$ on $\mathbb{R}$ defined by $a * b = a + b - ab$.
   (c) the operation $*$ on $\mathbb{R}$ defined by $a * b = a + b + ab$.
   (d) None of above.

2. What is the order of cyclic subgroup of the group C^* of nonzero complex numbers under multiplication generated by $\frac{1+i}{\sqrt{2}}$.
   (a) 2
   (b) 4
   (c) 8
   (d) 16

3. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, where $n \geq 2$. For $k \leq n$, let $E = \{v_1, v_2, \cdots, v_k\} \subset \mathbb{R}^n$ and $F = \{Tv_1, Tv_2, \cdots, Tv_k\}$. Then
   - (a) If E is linearly independent, then F is linearly independent
   - (b) If F is linearly independent, then E is linearly independen
   - (c) If E is linearly independent, then F is linearly dependen
   - (d) If F is linearly independent, then E is linearly dependen

4. For $n \neq m$, $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^n$ be linear transformations such that T_1T_2 is bijective. Then
   - (a) rank (T_1)=n and rank (T_2)=m
   - (b) rank (T_1)=m and rank (T_2)=n
   - (c) rank (T_1)=n and rank (T_2)=n
   - (d) rank (T_1)=m and rank (T_2)=m

5. Let A and B be open sets in $\mathbb{R}$. Then $A \cup B$ is
   - (a) an open set in $\mathbb{R}$.
   - (b) a closed set in $\mathbb{R}$.
   - (c) both open and closed in $\mathbb{R}$.
   - (d) Neither open nor closed in $\mathbb{R}$.

6. Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   Then
   - (a) $f'(x)$ is continuous at $x = 0$
   - (b) $f''(x)$ is continuous at $x = 0$
   - (c) $f'(0)$ exists
   - (d) $f''(0)$ exists

7. The value of the integral $\int_0^{\sqrt{\frac{\pi}{2}}} x \cos x^2 \, dx$ is equal to
   - (a) 1
   - (b) $\frac{1}{3}$
   - (c) $\frac{1}{2}$
   - (d) None of the above.

8. $\lim_{(x,y)\to(0,0)} \frac{tan^{-1}\ (2x+4y)}{sin^{-1}\ (x+2y)} =$
   - (a) 0
   - (b) 1
   - (c) 2
   - (d) Does not exist

9. The initial value problem $y' = x^2 + y^{-2/3}$ with $y(0) = 0$ has/have
   - (a) unique Solution
   - (b) no solution
   - (c) more than one solution
   - (d) exactly two solutions

10. $\int_0^1 \int_x^1 sin(y^2) dy dx =$
   - (a) $\frac{1+cos1}{2}$
   - (b) $1 - cos1$
   - (c) $1 + cos1$
   - (d) $\frac{1-cos1}{2}$

11. Let G is a non-abelian group of order 343. what is the order of its centre?
   - (a) 7
   - (b) 0
   - (c) 49
   - (d) 343.

12. The number of elements in a non-commutative group has at least
   - (a) 4
   - (b) 5
   - (c) 6
   - (d) None of above.

13. Given that the matrix $\begin{pmatrix} \alpha & 1 \\ 2 & 3 \end{pmatrix}$ has 1 as an eigenvalue. Then its trace and its determinant are

(a) 5, 4

(b) 4, 5

(c) 6, 4

(d) 4, 6

14. Let A be a 3×3 matrix with trace(A)=3 and det (A)=2. If 1 is an eigenvalue of A, then the eigenvalues of the matrix $A^2 - 2I$ are

(a) $1,\ 2(i-1),\ -2(i+1)$

(b) $-1,\ 2(i-1),\ 2(i+1)$

(c) $1,\ 2(i+1),\ -2(i+1)$

(d) $-1,\ 2(i-1),\ -2(i+1)$

15. The radius of the convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n \cdot x^n}{n!}$ is

(a) 1

(b) $\frac{1}{2}$

(c) e

(d) e^{-1}

16. Let $f(x)$ be the sum of the power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$ for some positive real number R. If $f(x) = f(-x)$ for all $x \in (-R, R)$, then

(a) $a_n \neq 0$ for all $n \geq 0$.

(b) $a_n = 0$ for all odd n.

(c) $a_n = 0$ for only finitely many values of n.

(d) None of the above

17. $f : \mathbb{R} \to \mathbb{R}$ is such that $f(0) = 0$ and $\left|\frac{df}{dx}(x)\right| \leq 5$ for all x. We can conclude that $f(1)$ is in

(a) $(5, 6)$

(b) $[-5, 5]$

(c) $(-\infty, -5) \cup (5, \infty)$

(d) $[-4, 4]$

18. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Then

(a) f is bounded above, but not bounded below.

(b) f is bounded above and below, but may not attains its bound.

(c) f is bounded above and below and f attains its bounds.

(d) f is differentiable at $x = 0$.

19. Let f be a differentiable function defined on $[0, 1]$. If $\xi \in (0, 1)$ is such that $f(x) < f(\xi) = f(0)$ for all $x \in (0, 1], x \neq \xi$, then

(a) $f'(\xi) = 0$ and $f'(0) = 0$

(b) $f'(\xi) = 0$ and $f'(0) \leq 0$

(c) $f'(\xi) > 0$ and $f'(0) \leq 0$

(d) $f'(\xi) = 0$ and $f'(0) > 0$

20. Let $f(x) = \int_1^{x^2} \cos t \, dt$. Then $f'(x)$ is equal to

(a) $\cos x^2$

(b) $2 \cos x^2$

(c) $2x \cos x^2$

(d) None of the above

21. Which of the following is true for the function $f : R^2 \to \mathbb{R}$ defined by $f(x,y) = \begin{cases} \dfrac{5}{1+2yx^2+e^{\frac{1}{x}}} + 3y^2, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise.} \end{cases}$

(a) $f(x,y)$ is continuous and differentiable at (0, 0).

(b) $f(x,y)$ is continuous, but not differentiable at (0, 0).

(c) $f(x,y)$ Continuous and has continuous partial derivatives at (0, 0).

(d) $f(x,y)$ is not continuous at (0, 0).

22. Which of the following function is continuous, but not differentiable

(a) $f(x,y) = \begin{cases} \dfrac{x^2 - 3x\sqrt{y}}{x^2+2y}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise.} \end{cases}$

(b) $f(x,y) = \begin{cases} tan^{-1}\dfrac{y}{x}, & \text{if } (x,y) \neq (0,1) \\ \frac{\pi}{2}, & \text{otherwise.} \end{cases}$

(c) $f(x,y) = \begin{cases} \dfrac{2x^3+3y^3}{3x^2-2y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise.} \end{cases}$

(d) None of the above.

23. If $u(x,y) = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, $0 < x, y < 1$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} =$

(a) $2u$

(b) $-\frac{1}{2}\cos\ u$

(c) $2\cos\ u$

(d) $-\frac{1}{2}\cot\ u$

24. Let $f(x,y) = 100 - 6x^2y$ be a function of two real variables x and y. Let R be a rectangular region defined by $R : 0 \leq x \leq 2, -1 \leq y \leq 1$. Then the value of the double integral

$$\iint_R f(x,y)\, dA$$

is

(a) 200

(b) 400.

(c) 350.

(d) 100

25. The initial value problem $(x^2 - 2x)y' = 2(x-1)y$, with $y(x_0) = y_0$ has more than one solution if and only if (Find conditions on x_0 and y_0.)

(a) $x_0(x_0-2) \neq 0$ and for all y_0

(b) $x_0(x_0-2) = 0$ and $y_0 \neq 0$

(c) $x_0(x_0-2) \neq 0$ and $y_0 = 0$

(d) $x_0(x_0-2) = 0$ and $y_0 = 0$

26. Let $y_1(x)$, $y_2(x)$ are two linearly independent solutions of $y'' + p(x)y' + q(x)y = 0,\ x \in I$. Then the functions $\phi(x) = \alpha y_1(x) + \beta y_2(x)$ and $\psi(x) = \gamma y_1(x) + \delta y_2(x)$ are two [2]

(a) linearly independent solution if $\alpha\delta = \beta\gamma$

(b) linearly independent solution if $\alpha\delta \neq \beta\gamma$

(c) linearly independent solution for all $\alpha, \delta, \beta, \gamma$

(d) linearly dependent solution for all $\alpha, \delta, \beta, \gamma$

27. If one solution of the corresponding homogeneous problem of the second order ODE

$$x(1-x)y''+2(1-2x)y'-2y = 0, \qquad x > 1$$

is $1/x$ then the second solution is

(a) $\dfrac{1}{1-x}$

(b) $\dfrac{1}{x(1-x)}$

(c) $\dfrac{1}{x}\ln(1-x)$

(d) $\dfrac{1}{x}\ln\dfrac{1}{1-x}$

28. If $\vec{F}(x, y, z) = (3 + 2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$ and let L be the curve

$$\vec{r}(t) = (e^t sint)\hat{i} + (e^t cost)\hat{j}, 0 \le t \le \pi.$$

Then $\int_L \vec{F} \cdot d\vec{r} =$

(a) $e^{-3\pi} + 1$

(b) $e^{-6\pi} + 2$

(c) $e^{6\pi} + 2$

(d) $e^{3\pi} + 1$

29. The flux of the vector field $\vec{F}(x, y, z) = \left(2\pi x + \frac{2x^2y^2}{\pi}\right)\hat{i} + \left(2\pi yx - \frac{4y}{\pi}\right)\hat{j}$ along the outward normal, across the ellipse $x^2 + 16y^2 = 4$ is equal to

(a) $4\pi^2 - 2$

(b) $2\pi^2 - 4$

(c) $\pi^2 - 2$

(d) 2π

30. The line integral of the vector field $\vec{F} = zx\hat{i} + xy\hat{j} + yz\hat{k}$ along the boundary of the triangle with vertices $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ oriented anticlockwise, when viewed from the point $(2, 2, 2)$ is

(a) $\frac{-1}{2}$

(b) -2

(c) $\frac{1}{2}$

(d) 2

Section B

1. Which of the following statements are true?

   (a) A finite group can not be expressed as the union of two of its proper subgroup.

   (b) If G is an abelian group then for all $a, b \in G$ all integers n, $(ab)^n = a^n b^n$.

   (c) The orders of the elements a and $x^{-1}ax$ are not same where a, x are any two elements of a group.

   (d) If G is a group of even order, then it has an element $a \neq e$ satisfying $a^2 = e$.

2. Pick out the true statements:

   (a) Let A and B be two arbitrary $n \times n$ matrices. Then $(A+B)^2 = A^2 + 2AB + B^2$.

   (b) There exist $n \times n$ matrices A and B such that $AB - BA = I$.

   (c) Let A and B be two arbitrary $n \times n$ matrices. If B is invertible, then $\text{tr}(A)=\text{tr}(B^{-1}AB)$.

   (d) Let A and B be two arbitrary $n \times n$ matrice. Then $|AB| = |BA|$.

3. Let A be a symmetric $n \times n$ matrix with real entries, which is positive semi-definite, i.e. $x^t Ax \geq 0$ for every (column) vector x. Pick out the true statements:

   (a) the eigenvalues of A are all non-negative .

   (b) A is invertible .

   (c) the principal minor Δ_k of A (i.e. the determinant of the $k \times k$ matrix obtained from the first k rows & first k columns of A) is non-negative for each $1 \leq k \leq n$.

   (d) Eigen vectors generates $\mathbb{R}^n$

4. Between two distinct real numbers, there exists/ exist

   (a) a unique rational number.

   (b) infinite number of rational numbers.

   (c) a unique irrational number.

   (d) infinite number of irrational numbers.

5. Let K be a subset of $\mathbb{R}$. Consider the following statements:

   (i) K is closed and bounded in $\mathbb{R}$.

   (ii) K is a compact set in $\mathbb{R}$.

   (iii) Every infinite subset of K has a limit point in K.

   Choose the correct option(s):

   (a) Statement (i) does not imply Statement (ii).

   (b) Statement (i) implies Statement (iii).

   (c) Statement (i) does not imply Statement (iii).

   (d) All statements are equivalent.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = (x-2)^{17}(x+5)^{24}$. Then

   (a) f have exactly three critical points in $\mathbb{R}$.

   (b) f have exactly two critical points in $\mathbb{R}$.

   (c) f have a local minimum at $x = 2$.

   (d) f has neither a local minimum nor a local maximum at $x = 2$

7. Which of the following function possesses partial derivatives $f_x(0,0), f_y(0,0)$, but partial derivatives are not continuous at $(0,0)$

   (a) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise.} \end{cases}$

   (b) $f(x,y) = \begin{cases} \dfrac{y(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise.} \end{cases}$

   (c) $f(x,y) = \begin{cases} \dfrac{xy(2x^2-3y^2)}{3x^2-2y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise.} \end{cases}$

   (d) $f(x,y) = \begin{cases} (2x+2y)\sin(\dfrac{1}{x+y}), & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise.} \end{cases}$

8. Let $f(x,y)$ and $g(x,y)$ be two real-valued continuous functions defined on a rectangular region $R : a \leq x \leq b, c \leq y \leq d$. Which of the following statements is/ are true?

   (a) $\iint_R (f(x,y) + g(x,y))dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$.

   (b) $\iint_R f(x,y)dA \geq \iint_R g(x,y)dA$ if $f(x,y) \geq g(x,y)$ on R.

   (c) $\iint_R f(x,y)dA \geq 0$.

   (d) $\iint_R kf(x,y)dA = |k| \iint_R f(x,y)dA$, where k is a real number.

9. The two functions $\ln x$ and $\ln x^3$ are

   (a) linearly independent in $(0,1)$

   (b) linearly independent in $(1,\infty)$

   (c) linearly dependent in $(0,\infty)$

   (d) linearly dependent in $(1,\infty)$

10. If $\vec{F}(x,y,z) = (2x+3yz)\hat{i} + (2y+3xz)\hat{j} + (2z+3xy)\hat{k}$ for $(x,y,z) \in \mathbb{R}^3$, then which among the following is (are) true?

    (a) $\nabla \times \vec{F} = \vec{0}$.

    (b) $\oint_C \vec{F}.d\vec{r} = 0$

    (c) There exists a scalar function $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla \cdot \vec{F} = \phi_{xx} + \phi_{yy} + \phi_{zz}$.

    (d) $\nabla \cdot \vec{F} = 0$

Section C

1. Let G be a group and $a \in G$ such that $O(a) = n$, then $O(a^k)$, where $k \in \mathrm{N}$ is __________________________.

2. The number of generators of the cyclic group of order 29 are __________________________.

3. Let A be a non-diagonal 2×2 matrix with complex entries such that $A = A^{-1}$. Then its characteristic and minimal polynomials are __________________________.

4. The matrix A has $(1,2,1)^t$ and $(1,1,0)^t$ as eigenvectors, both with eigenvalue 7, and its trace is 2. Then the value of the determinant of A is __________________________.

5. The set of limit points of the set $S = \{x \in \mathbb{R} : 1 < x < 3\}$ is __________________________.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $|f(x) - f(y)| \geq |x - y|$ for every $x, y \in \mathbb{R}$.

   TRUE/FALSE: f is one-one. __________________________.

7. Let $\phi(x,y,z) = 3y^2 + 3yz$ for $(x,y,z) \in \mathbb{R}^3$. Then the absolute value of the directional derivative of ϕ in the direction of the line $\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z}{-2}$ at the point $(1,-2,1)$ is __________________________.

8. The area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant is equal to __________________________.

9. Let a function $f(x,y)$ be continuous and possess first and second order partial derivatives at a point $P(a,b)$. If $P(a,b)$ is a critical point, then the point P is a point of relative maxima if __________________________.

10. Let $p(x)$, $q(x)$, $r(x)$ are continuous functions on an open interval I. Further, suppose $y_1(x)$, $y_2(x)$ are any two solutions of the linear non-homogeneous equation $(r(x) \neq 0)$

$$y'' + p(x)y' + q(x)y = r(x), \qquad x \in I. \tag{1}$$

Then $ay_1 + by_2$ is also its solution if and only if __________________________.

11. **TRUE/FALSE:** $\mathbf{Z_4}$ is isomorphic to $\mathbf{Z_2} \times \mathbf{Z_2}$. __________________________.

12. Let $a_1, a_2, \cdots, a_n \in \mathbb{R}$. The value of the determinant
$$\begin{vmatrix} 1+a_1 & a_2 & \cdots & a_n \\ a_1 & 1+a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & 1+a_n \end{vmatrix}$$
is __________________________.

13. Using second degree Taylor's polynomial of $f(x) = x^{1/3}$ around $x = 1$, the approximate value of $2^{1/3}$ is __________________________.

14. Let f be a real-valued function defined on the closed interval $[a,b]$ which is also continuous on this interval. Let $F(x) = \int_a^x f(t)\,dt$ for all $x \in [a,b]$. Then $F'(x) =$ __________________________ for all $x \in (a,b)$. If the function G is any antiderivative of f on $[a,b]$, then $\int_a^b f(x)\,dx =$ __________________________.

15. The radius of the convergence of the power series $1 + 2 \cdot 3x + 3 \cdot 3^2x^2 + \cdots + n \cdot 3^{n-1}x^{n-1} + \cdots$ is __________________________. Let $f(x)$ be the sum of the above power series defined on the interval $(-R,R)$, where R is the radius of the convergence of the above series. Then the value of $\int_0^{1/4} f(x)dx$ is __________________________.

16. The minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$ is ____________________.

17. The volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle $R : 0 \leq x \leq 1, 0 \leq y \leq 2$ is equal to ____________________. The value of the integral $\iint_R e^{x^2+y^2}\, dy\, dx$, where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$, is equal to ____________________.

18. The orthogonal trajectories of the curve $e^x \sin y = c$ is ____________________.

19. Let C be the boundary of the region enclosed by $y = x^2,\ y = x + 2$ and $x = 0$. Then the value of the line integral

$$\oint_C (xy - y^2)dx - x^3 dy,$$

where C is traversed in the counterclockwise direction, is ____________________.

20. Let S be the closed surface forming the boundary of the region V bounded by $x^2 + y^2 = 3,\ z = 0,\ z = 6$. A vector field $\vec{F}$ is defined over V with $\nabla \cdot \vec{F} = 2y + z + 1$. Then the value of

$$\frac{1}{\pi} \iint_S \vec{F} \cdot \hat{n} dS,$$

where $\hat{n}$ is the unit outward drawn normal to the surface S, is ____________________.